Position Auctions

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December 2005
Revised: March 29, 2006

Abstract

I analyze the equilibria of a game based on the ad auction used by Google and Yahoo. This auction is closely related to the assignment game studied by Shapley-Shubik, Demange-Gale-Sotomayer and Roth-Sotomayer. However, due to the special structure of preferences, the equilibria of the ad auction can be calculated explicitly and some known results can be sharpened. I provide some empirical evidence that the Nash equilibria of the position auction describe the basic properties of the prices observed in Google's ad auction reasonably accurately.

[*]I received many helpful comments from Marc Berndl, John Lamping, Amit Patel, Rob Shillingsburg, Diane Tang, and Eric Veach. I am particularly grateful to Meredith Goldsmith for her close reading of the paper, which improved the exposition significantly. I also thank Jonathan Rosenberg for allowing me to publish these results. Email contact: hal@sims.berkeley.edu

I consider the problem of assigning agents $a = 1, \ldots, A$ to slots $s = 1, \ldots, S$ where agent a's valuation for slot s is given by $u_{as} = v_a x_s$. We number the slots so that $x_1 > x_2 > \cdots > x_S$ so that all agents agree on their ordering of the slots, though each agent may value them differently. We also set $x_s = 0$ for all $s > S$ and assume that the number of agents is at least equal to the number of slots plus 1.

This problem is motivated by the ad auctions used by Google and Yahoo. In these auctions the agents are advertisers and the slots are positions on a web page. Higher positions receive more clicks, so x_s can be interpreted as the clickthrough rate for slot s. The value $v_a > 0$ can be interpreted as the expected profit per click so $u_{as} = v_a x_s$ indicates the expected profit to advertiser a from appearing in slot s. The slots are sold via an auction. Each agent bids an amount b_a, with the best clickthrough rate being assigned to the agent with the highest bid, the second-best slot to the agent with the second highest bid, and so on. Renumbering the agents if necessary, let v_s be the value per click of the agent assigned to slot s. The price agent s faces is the bid of the agent immediately below him, so $p_s = b_{s+1}$. Hence the net profit that agent a can expect to make if he acquires slot s is $(v_a - p_s)x_s = (v_a - b_{s+1})x_s$.

Google's ad auction generated about $1.5 billion in Q3 2005 so its financial success alone makes it worthy of study. We will also see that position auctions have a nice mathematical structure and a strong relationship to existing literature on two-sided matching models. Edelman et al. [2005] independently examine these auctions and develop related results. However, our treatments are somewhat different and I also add some empirical analysis.

1 Nash equilibrium of position auction

Consider Table 1 which depicts the positions, values, bids and payment associated with an auction with $S = 4$ available slots. We know that $x_s > x_{s+1}$ by assumption and that $b_s > b_{s+1}$ by the rules of the auction.

If agent 3 wanted to move up by one position, it would have to bid at least b_2, the bid of agent 2. But if agent 2 wanted to move down by one position it would only have bid at least $b_4 = p_3$, the bid of agent 4. We see that to move to a higher slot you have to beat the *bid* that the agent who currently occupies that slot is making; to move to a lower slot you only have to beat the *price* that the agent who currently occupies that slot is paying.

We assume that the agents choose their bids to maximize their expected

Position	Value	Bid	Price	CTR
1	v_1	b_1	$p_1 = b_2$	x_1
2	v_2	b_2	$p_2 = b_3$	x_2
3	v_3	b_3	$p_3 = b_4$	x_3
4	v_4	b_4	$p_4 = b_5$	x_4
5	v_5	b_5	0	0

Table 1: Bidding for position

profit given the bids made by the other agents. In equilibrium, each agent should prefer his current slot to any alternative slot, which motivates the following definition.

Definition 1 *A Nash equilibrium (NE) is a set of prices such that*

$$(v_s - p_s)x_s \geq (v_s - p_t)x_t \text{ for } t > s \tag{1}$$
$$(v_s - p_s)x_s \geq (v_s - p_{t-1})x_t \text{ for } t < s \tag{2}$$

where $p_t = b_{t+1}$.

Note that if the inequalities are strict and an agent changes his bid slightly it won't affect his position or payment, so there will generally be a range of bids and prices that satisfy these inequalities. Also note that these inequalities are linear in the prices. Hence, given (v_s) and (x_s) we can use a simple linear program to solve for the maximum and minimum equilibrium revenue attainable by the auction.

The analysis of the position auction is much simplified by examining a particular subset of Nash equilibria.

Definition 2 *A symmetric Nash equilibrium (SNE) is a set of prices such that*

$$(v_s - p_s)x_s \geq (v_s - p_t)x_t \text{ for all } t \text{ and } s.$$

Equivalently,

$$v_s(x_s - x_t) \geq p_s x_s - p_t x_t \text{ for all } t \text{ and } s.$$

Note that the inequalities characterizing an SNE are the same as the inequalities characterizing an NE for $t > s$, inequalities (1), but expressed for all s and t. I will show in a series of short arguments that the symmetric Nash equilibria form a well-behaved subset of the Nash equilibria that can be calculated explicitly.[1]

Fact 1 (Non-negative surplus) *In an SNE* $v_s \geq p_s$.

Proof. Using the inequalities defining an SNE,

$$(v_s - p_s)x_s \geq (v_{S+1} - p_{S+1})x_{S+1} = 0,$$

since $x_{S+1} = 0$. □

Fact 2 (Monotone values) *In an SNE,* $v_{s-1} \geq v_s$ *for all* s.

Proof. By definition of SNE we have

$$\begin{aligned} v_t(x_t - x_s) &\geq p_t x_t - p_s x_s \quad (3) \\ v_s(x_s - x_t) &\geq p_s x_s - p_t x_t \quad (4) \end{aligned}$$

Adding these two inequalities gives us

$$(v_t - v_s)(x_t - x_s) \geq 0,$$

which shows that (v_t) and (x_t) must be ordered the same way. □

Fact 3 (Monotone prices) *In an SNE,* $p_{s-1}x_{s-1} > p_s x_s$ *and* $p_{s-1} > p_s$ *for all* s.

Proof. By definition of SNE we have

$$(v_s - p_s)x_s \geq (v_s - p_{s-1})x_{s-1},$$

which can be rearranged to give

$$p_{s-1}x_{s-1} \geq p_s x_s + v_s(x_{s-1} - x_s) > p_s x_s.$$

This proves the first part.

[1] It is worth observing that the set of prices (p_s) for an SNE comprise a market equilibrium for the assignment problem; see e.g. Gale [1960]. We explore this connection further in section 4.

To prove the second part, note that since $v_s \geq p_s$ by Fact 1, we can apply (1) to find

$$p_{s-1}x_{s-1} > p_s x_s + v_s(x_{s-1} - x_s) \geq p_s x_s + p_s(x_{s-1} - x_s) = p_s x_{s-1}. \square$$

Earlier we motivated the definition of NE and SNE by assuming $p_{t-1} > p_t$. The Monotone Prices fact shows that these inequalities follow from directly from the definition of SNE, so the earlier assumption was in fact redundant.

Fact 4 (NE $\supset$ SNE) *If a set of prices is an SNE it is an NE.*

Proof. Since $p_{t-1} > p_t$,

$$(v_s - p_s)x_s \geq (v_s - p_t)x_t \geq (v_s - p_{t-1})x_t.$$

for all s and t. $\square$

The reason that the set of symmetric Nash equilibria is interesting is that it is only necessary to verify the inequalities for one step up or down in order to verify that the entire set of inequalities is satisfied.

Fact 5 (One step solution) *If a set of bids satisfies the symmetric Nash equilibria inequalities for $s+1$ and $s-1$, then it satisfies these inequalities for all s.*

Proof. I give a proof by example. Suppose that the SNE relations hold for slots 1 and 2 and for slots 2 and 3; we need to show it holds for 1 and 3. Writing out the condition and using the fact that $v_1 \geq v_2$,

$$\begin{aligned} v_2(x_1 - x_2) \geq p_1x_1 - p_2x_2 &\to v_1(x_1 - x_2) \geq p_1x_1 - p_2x_2 \\ v_2(x_2 - x_3) \geq p_2x_2 - p_3x_3 &\to v_1(x_2 - x_3) \geq p_2x_2 - p_3x_3 \end{aligned}$$

Adding up the left and right columns,

$$v_1(x_1 - x_3) \geq p_1x_1 - p_3x_3,$$

as was to be shown. The argument going the other direction is similar. $\square$

These facts allow us to provide an explicit characterization of equilibrium prices and bids. Since the agent in position s does not want to move down one slot:

$$(v_s - p_s)x_s \geq (v_s - p_{s+1})x_{s+1}$$

Since the agent in position $s+1$ does not want to move up one slot:

$$(v_{s+1} - p_{s+1})x_{s+1} \geq (v_{s+1} - p_s)x_s.$$

Putting these two inequalities together we see:

$$v_s(x_s - x_{s+1}) + p_{s+1}x_{s+1} \geq p_s x_s \geq v_{s+1}(x_s - x_{s+1}) + p_{s+1}x_{s+1}. \tag{5}$$

Recalling that $p_s = b_{s+1}$ we can also write these inequalities as:

$$v_{s-1}(x_{s-1} - x_s) + b_{s+1}x_s \geq b_s x_{s-1} \geq v_s(x_{s-1} - x_s) + b_{s+1}x_s. \tag{6}$$

Defining $\alpha_s = x_s/x_{s-1} < 1$, we can also write the inequalities as:

$$v_{s-1}(1 - \alpha_s) + b_{s+1}\alpha_s \geq b_s \geq v_s(1 - \alpha_s) + b_{s+1}\alpha_s. \tag{7}$$

The equivalent conditions (5)-(7) show that in equilibrium each agent's bid is bounded above and below by a convex combination of the bid of the agent below him and a value—either his own value or the value of the agent immediately above him. The (pure strategy) Nash equilibria can be found simply by recursively choosing a sequence of bids that satisfy these inequalities.

We can examine the limiting cases by choosing the upper and lower bounds in inequalities (6). The recursions then become

$$b_s^U x_{s-1} = v_{s-1}(x_{s-1} - x_s) + b_{s+1}x_s \tag{8}$$

$$b_s^L x_{s-1} = v_s(x_{s-1} - x_s) + b_{s+1}x_s \tag{9}$$

The solution to these recursions are:

$$b_s^U x_{s-1} = \sum_{t \geq s} v_{t-1}(x_{t-1} - x_t). \tag{10}$$

$$b_s^L x_{s-1} = \sum_{t \geq s} v_t(x_{t-1} - x_t). \tag{11}$$

The starting values for the recursions follow from the fact that there are only S positions, so that $x_s = 0$ for $s > S$. Writing out the lower bound on the bid for $s = S+1$, we have

$$\begin{aligned} b_{S+1}^L x_S &= v_{S+1}(x_S - x_{S+1}) \\ &= v_{S+1}x_S \end{aligned}$$

so that it is optimal for the first excluded bidder to bid his value. This is has the same logic as the usual Vickrey auction. If you are excluded, then bidding lower than your value is pointless, but if you do happen to be shown (e.g., because one of the higher bidders drops out) you will make a profit.

1.1 Logic of the bounds

Of course, any bid in the range described by (5)-(7) is an SNE and hence an NE bid, but perhaps there are reasons why bidding at one end of the upper or lower bounds might be particularly attractive.

Suppose that I am in position s making a profit of $(v_s - b_{s+1})x_s$. In Nash equilibrium my bid is optimal given my beliefs about the bids of the other agents, but I can vary my bid in range specified by (6) without changing my payments or position.

What is the highest bid I can set so that if I happen to exceed the bid of the agent above me and I move up by one slot, I am sure to make at make at least as much profit as I make now?

The worst case is where I just beat the advertiser above me by a tiny amount and end up paying my bid, b_s, minus a tiny amount. Hence the breakeven case satisfies the equation

$$\begin{aligned} \text{worst case profit moving up} &= \text{profit now} \\ (v_s - b_s^*)x_{s-1} &= (v_s - b_{s+1})x_s. \end{aligned}$$

Solving for b_s^* gives us

$$b_s^* x_{s-1} = v_s(x_{s-1} - x_s) + b_{s+1}x_s,$$

which is the lower-bound recursion, (9).

Alternatively, we can think defensively. If I set my bid too high, I will squeeze the profit of the player ahead of me so much that he might prefer to move down to my position. The highest breakeven bid that would not induce the agent above me to move down is

$$\text{his profit now} = \text{how much he would make in my position} \quad (12)$$

$$(v_{s-1} - b_s^*)x_{s-1} = (v_{s-1} - b_{s+1})x_s. \quad (13)$$

Solving for b_s^* gives us

$$b_s^* x_{s-1} = v_{s-1}(x_{s-1} - x_s) + b_{s+1}x_s,$$

which is the upper-bound recursion, (8).

As a matter of practice, it seems to me that the first argument is compelling. Even though any bid in the range (5) is a Nash bid, one might argue that setting that bid so that I make a profit if I move up in the ranking is a reasonable strategy.

2 NE revenue and SNE revenue

Summing equations (10) and (11) over $s = 1, \ldots S$ gives us upper and lower bounds on total revenue in an SNE. If the number of slots $S = 3$, for example, the lower and upper bounds are given by

$$\begin{aligned} R^L &= v_2(x_1 - x_2) + 2v_3(x_2 - x_3) + 3v_4 x_3 \\ R^U &= v_1(x_1 - x_2) + 2v_2(x_2 - x_3) + 3v_3 x_3. \end{aligned}$$

How do these bounds relate to the bounds for the NE calculated by the linear programming problems alluded to earlier?

Since the set of NE contains the set of SNEs, one might speculate that the maximum and minimum revenues are larger and smaller, respectively, than the SNE maximum and minimum revenue. This is half right: it turns out that the upper bound for the SNE revenue is the *same* as the maximum revenue for the NE, while the lower bound on revenue from the NE is generally *less than* the revenue bound for the SNE.

Fact 6 *The maximum revenue NE yields the same revenue as the upper recursive solution to the SNE.*

Proof. Let (p_s^N) be the prices associated with the maximum revenue Nash equilibrium and let (p_s^U) be the prices that solve the upper recursion for the SNE. Since NE $\supset$ SNE, the revenue associated with (p_s^N) must be at least as large as the revenue associated with (p_s^U).

From the definition of an NE, (1), we have:

$$p_s^N x_s \leq p_{s+1}^N x_{s+1} + v_s(x_s - x_{s+1}).$$

From the definition of the upper-bound recursion, (8), we have:

$$p_s^U x_s = p_{s+1}^U x_{s+1} + v_s(x_s - x_{s+1}).$$

The recursions start at $s = S$. Since $x_{S+1} = 0$ we have

$$p_S^N \leq v_S = p_S^U.$$

It follows by inspecting the recursions immediately above that $p_s^U \geq p_s^N$ for all s. Hence the maximum revenue from the SNE is at least as large as the maximum revenue from the NE, implying that the revenue must be equal. $\square$

It is easy to construct examples where the minimum revenue NE has less revenue than the solution to the lower recursion for the SNE; this is not surprising since the set of inequalities defining the NE strictly contains the set of inequalities defining the SNE. So we have the general relations:

$$\begin{aligned} \text{maximum revenue NE} &= \text{value of upper recursion of SNE} \geq \\ \text{value of lower recursion of SNE} &\geq \text{min revenue NE} \end{aligned}$$

with the inequalities being strict except in degenerate cases.

3 Previous literature

We have already mentioned the recent analysis of Edelman et al. [2005]. However, there is a much older literature that is closely related to the position auction problem.

Shapley and Shubik [1972] describe an assignment game in which agents are assigned objects with at most one object being assigned to an agent. Mathematically, let agent a's evaluation of object s be given by u_{as}. The assignment problem asks for the assignment of objects to agents that maximizes value. This problem can be solved by linear programming or by other algorithms.

It turns out that an optimal assignment can be decentralized by means of price mechanism. That is, at an optimal assignment there will exist a set of numbers (p_a), interpretable as the price of the object assigned to agent a, such that:

$$u_{as} - p_a \geq u_{as} - p_b \qquad \text{for all } a \text{ and } b.$$

Hence at the prices (p_a) each agent would weakly prefer the object assigned to him over any other object.

Comparing this to the definition of the symmetric Nash inequalities, we see that the definitions are the same with $u_{as} = v_a x_s$ and $p_a = b_{a+1} x_s$. Hence, the position auction game we have described is simply a competitive equilibrium of an assignment game that has a special structure for utility. However, the special structure is particularly natural in this context. In particular, we can explicitly solve for the largest and smallest competitive equilibrium due to the special structure of u_{as}.

Demange et al. [1986] construct an auction that determines a competitive equilibrium. However, the auction they construct is quite different from the position auction. Roth and Sotomayor [1990], Chapter 8, contains an unified treatment of these results. Several papers have developed auctions that yield competitive equilibria for the assignment game; see Bikhchandani and Ostroy [2006] for a recent survey.

4 Incentives

We have seen that the optimal bids in the position auction will in general depend on the bids made by other agents. One might well ask if there is a way to find another auction structure for which agent a's optimal bid depends only on its value. Is it possible to find an auction form that has a dominant strategy equilibrium? Demange and Gale [1985] show the answer is "yes," using a variation on the Hungarian algorithm for the assignment problem.

If we relax our conception of what an auction is, we can apply the well-known Vickrey-Clarke-Groves mechanism to this problem. Leonard [1983] describes this for the general case, but the VCG mechanism takes a particularly simple form for the special case we are considering here.

Let us recall the basic structure of the VCG mechanism. Suppose a central authority is going to choose some outcome z so as to maximize the sum of the reported utilities of agents $a = 1, \ldots, A$. Let agent a's true utility function be denoted by $u_a(\cdot)$ and its reported utility function by $r_a(\cdot)$.

In order to align incentives, the center announces it will pay each agent the sum of the utilities reported by the other agents at the utility-maximizing outcome. Thus the center announces it is going to maximize

$$r_a(z) + \sum_{b \neq a} r_b(z)$$

while agent a cares about

$$u_a(z) + \sum_{b \neq a} r_b(z).$$

It is easy to see that in order to maximize its own payoff, agent a will want to report its true utility function, that is, set $r_a(\cdot) = u_a(\cdot)$, since this ensures that the center optimizes exactly what agent a wants it to maximize.

We can reduce the size of the sidepayments by subtracting an amount from agent a that does not depend on its report. A convenient choice in this respect is the maximized sum of reported utilities omitting agent a's report. Hence the final payoff to agent a becomes

$$u_a(z) + \sum_{b \neq a} r_b(z) - \max_y \sum_{b \neq a} r_b(y).$$

The payment made by agent a can be interpreted as the harm that its presence imposes on the other agents: that is the difference between what they get when agent a is present and what they would get if agent a were absent.

In the context of assigning agents to positions, if agent $s-1$ is omitted, each agent below agent $s-1$ will move up one position while agents above $s-1$ are unaffected. Hence the payment that agent $s-1$ must make is

$$\text{VCG payment of agent } s-1 = \sum_{t \geq s} r_t(x_{t-1} - x_t), \tag{14}$$

where r_t is the reported value of agent t. In the dominant strategy VCG equilibrium, each agent t will announce $r_t = v_t$, so

$$\text{equilibrium VCG payment of agent } s-1 = \sum_{t \geq s} v_t(x_{t-1} - x_t). \tag{15}$$

Comparing this to expression (11) this is easily seen to be the same as the lower bound for the symmetric Nash equilibria.

This relationship is true in general, even for arbitrary u_{as}. Demange and Gale [1985] show that the best (i.e., lowest cost) equilibrium for the buyers in the competitive equilibrium for the assignment problem is that given by the VCG mechanism. See Roth and Sotomayor [1990] for a detailed development of this theory, and Bikhchandani and Ostroy [2006] for a recent survey of related results.

5 Bounds on values

Returning to the symmetric Nash equilibrium analysis, it is possible to derive useful bounds on the unobserved values of the agents by using the observed equilibrium prices.

Let $p_s = b_{s+1}$ be the equilibrium price paid by agent s in a particular symmetric Nash (or competitive) equilibrium. Then we must have:

$$(v_s - p_s)x_s \geq (v_s - p_t)x_t.$$

Rearranging this we have

$$v_s(x_s - x_t) \geq p_s x_s - p_t x_t.$$

Dividing by $x_s - x_t$ and remembering that the sense of the inequality is reversed when $x_s < x_t$, we have

$$\min_{t>s} \frac{p_s x_s - p_t x_t}{x_s - x_t} \geq v_s \geq \max_{t<s} \frac{p_s x_s - p_t x_t}{x_s - x_t}.$$

Furthermore, we know from Fact 5 that the max and the min are attained at the neighboring positions, so we can write

$$\frac{p_{s-1} x_{s-1} - p_s x_s}{x_{s-1} - x_s} \geq v_s \geq \frac{p_s x_s - p_{s+1} x_{s+1}}{x_s - x_{s+1}}$$

These inequalities have a nice interpretation: the ratios are simply the incremental cost of moving up or down one position.

We can recursively apply these inequalities to write

$$v_1 \geq \frac{p_1 x_1 - p_2 x_2}{x_1 - x_2} \geq \tag{16}$$

$$v_2 \geq \frac{p_2 x_2 - p_3 x_3}{x_2 - x_3} \geq \tag{17}$$

$$\vdots$$

$$v_S \geq p_S \tag{18}$$

This shows that the incremental costs must decrease as we move to lower positions. This observation has three important implications.

1. The inequalities give an observable necessary condition for a the existence of a pure strategy Nash equilibrium, namely, that each of the intervals be non-empty. The conditions are also sufficient in that if the intervals are non-empty, we can find a set of values that are consistent with equilibrium.

2. The inequalities also yield simple bidding rule for the agents: if your value exceeds the marginal cost of moving up a position, then bid higher, stopping when this no longer is true.

3. Finally the inequalities motivate the following intuitive characterization of SNE: the marginal cost of a click must increase as you move to higher positions. Why? Because if it ever decreased, there would be an advertiser who passed up cheap clicks in order to purchase expensive ones.

We can also do the same calculations for the NE inequalities which yields:

$$\min_{t>s} \frac{p_s x_s - p_{t-1} x_t}{x_s - x_t} \geq v_s \geq \max_{t<s} \frac{p_s x_s - p_t x_t}{x_s - x_t}. \tag{19}$$

Note that the upper bounds for the NE (for $t > s$) are looser than for the SNE and that they now involve the entire set of bids, not just the neighboring bids.

6 Geometric interpretation

Figure 1 shows the clickthrough rates x_s on the horizontal axes and SNE expenditure $p_s x_s = b_{s+1} x_s$ on the vertical axis. We refer to this graph as the *expenditure profile.* The slope of the line segments connecting each vertex are the marginal costs described in the previous section which we have shown must bound the agents' values. According to the above discussion, if the observed choices are an SNE, this graph must be an increasing, convex function.

The profit accruing to agent s is $\pi_s = v_s x_s - p_s x_s$. Hence the iso-profit lines are given by $p_s x_s = v_s x_s - \pi_s$, which are straight lines with slope v_s and vertical intercept of $-\pi_s$. A profit-maximizing bidder wants to choose that position which has the lowest associated profit, as shown in Figure 1. The range of values associated with equilibrium are simply the range of slopes of the supporting hyperplanes at each point.

This diagram also can be used to illustrate the construction of the SNE using the recursive solution outlined earlier. Suppose that there are 3 slots and we are given four values. Since we know that $p_3 = v_4$ from the boundary condition for the lower recursion, we draw a line with slope v_4 connecting the points $(0, 0)$ and $(x_3, v_4 x_3)$. Next draw a line with slope of v_3 starting at $(x_3, v_4 x_3)$. The value of this line at x_2 will be $v_4 x_3 + v_3(x_2 - x_3)$, which is exactly the lower recursion. Continuing in this way traces out the equilibrium expenditure profile.

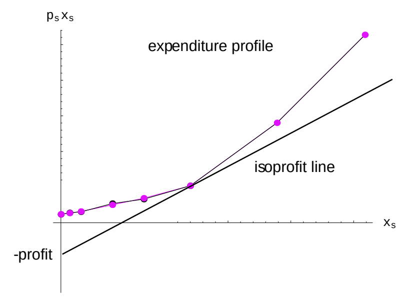

Figure 1: Expenditure profile for SNE.

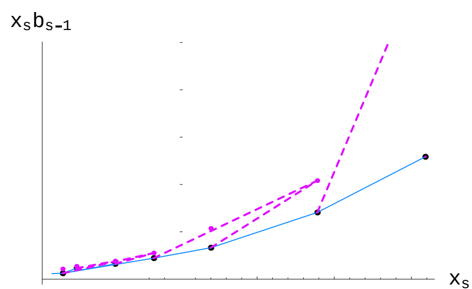

Figure 2: Expenditure profile for NE and SNE.

We can also illustrate the NE bounds using the same sort of diagram. The lower bounds on Figure 2 shows the the SNE bounds, along with the NE bounds from inequalities (19). For the NE, the lower bounds are the same, while the upper bounds are looser (steeper) than the SNE.

7 Applications to ad auctions

Up until now we have described the abstract strategic structure of the position auction. In order to apply this to the actual ad auction used by Google, we have to add some refinements.

Google actually ranks the ads by the product of a measurement of ad quality and advertiser bid, rather than just the bid alone.[2] We assume that the observed clickthrough rate for advertiser a in position s is the product of

[2] See `http://services.google.com/awp/en_us/breeze/5310/index.html`.

this "quality effect" e_s, and a "position effect," x_s. Letting z_s be advertiser s's observed clickthrough rate, we write $z_s = e_s x_s$.

Advertisers are ordered by $e_s b_s$ and each advertiser pays the minimum amount that is necessary to retain his position. Let q_{st} be the amount that advertiser s would need to pay to be in position t. By construction we have

$$q_{st} e_s = b_{t+1} e_{t+1}.$$

Solving for q_{st} we have

$$q_{st} = b_{t+1} e_{t+1} / e_s. \tag{20}$$

Nash equilibrium requires that each agent prefer his position to any other position, recognizing that the cost and clickthrough rate of the other position depends on his ad quality:

$$(v_s - q_{ss}) e_s x_s \geq (v_s - q_{st}) e_s x_t.$$

Substituting (20) into this expression and simplifying we have

$$(e_s v_s - b_{s+1} e_{s+1}) x_s \geq (e_s v_s - b_{t+1} e_{t+1}) x_t.$$

Letting $p_s = b_{s+1} e_{s+1}$ and $p_t = b_{t+1} e_{t+1}$ gives us

$$(e_s v_s - p_s) x_s \geq (e_s v_s - p_t) x_t.$$

We can now apply the same logic used in (16–18) to give us

$$\begin{aligned}
e_1 v_1 &\geq \frac{p_1 x_1 - p_2 x_2}{x_1 - x_2} \geq && (21)\\
e_2 v_2 &\geq \frac{p_2 x_2 - p_3 x_3}{x_2 - x_3} \geq && (22)\\
&\vdots \\
e_S v_S &\geq p_S. && (23)
\end{aligned}$$

These are the testable inequalities implied by the symmetric Nash equilibrium model.

Finally, we also have to mention the case of "non-fully sold pages" which are auctions where the number of ads displayed on the right-hand side is fewer than 8. In this case, the bottom ad on the page pays a reserve price which is currently set at 5 cents.

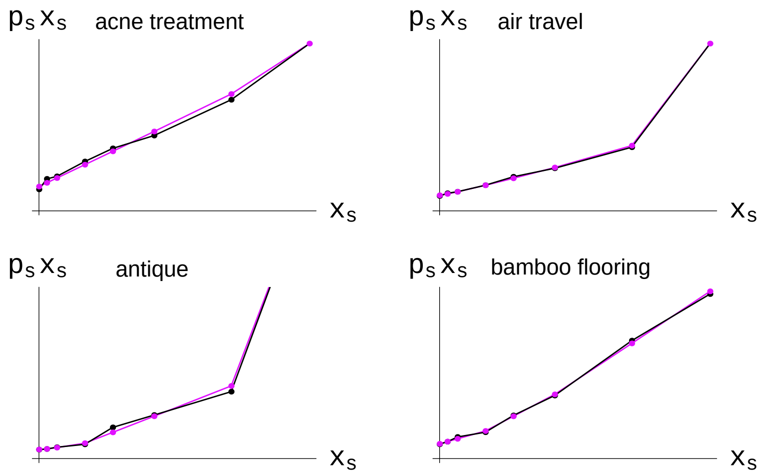

Figure 3: Examples of data and fits.

8 Empirical analysis

Given a set of position effects, quality effects, and bids we can plot x_t versus expenditure $b_t x_t$ and see if this expenditure profile is increasing and convex. It turns out that this often is true. If the graph is not increasing and convex, we can ask for a perturbation of the data that does exhibit these properties.

The question is, what to perturb? The natural variable to perturb is the ad quality, e_s, since this is the most difficult variable for the advertisers to observe and thus has the most associated uncertainty. Let $(d_s e_s)$ be the value of the perturbed ad quality where (d_s) is a set of multipliers indicating how much each ad quality needs to be perturbed to satisfy the Nash inequalities (21−23). Since the prices p_s are linear functions of e_s, we can also think of the perturbations as applying to the prices.

This model motivates the following quadratic programming problem: choose the perturbations (d_s) to be as close as possible to 1 (in terms of squared error) constrained by the requirement that the SNE inequalities given in (21−23) are satisfied.

Explicitly, the quadratic programming problem is

$$\min_d \sum_s (d_s - 1)^2$$

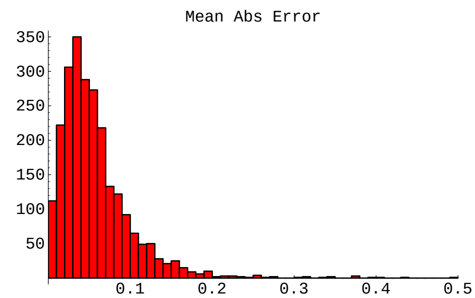

Figure 4: Distribution of mean absolute deviations.

$$\text{such that } \frac{d_{s-1}p_{s-1}x_{s-1} - d_s p_s x_s}{(x_{s-1} - x_s)} \geq \frac{d_s p_s x_s - d_{s+1}p_{s+1}x_{s+1}}{(x_s - x_{s+1})}.$$

Since the constraints are linear in (d_s), this is a simple quadratic programming problem which can be easily solved by standard methods. This minimal perturbation calculation can be given a statistical interpretation; see Varian [1985]. However, we do not pursue the details of this interpretation here.

Figure 3 shows some examples of expenditure profiles using the actual data along with the best fitting increasing, convex relationship. (The numeric values on the axes have been removed since this analysis is based on proprietary data.)

It can be seen that the general shape of the expenditure profile tends to be increasing and convex as the theory predicts. Furthermore, it often rather flat at least in positions 3-8. One explanation for the increased expenditure on positions 1 and 2 on the right-hand side is that Google will promote ads in these slots to the top-of-page position under certain conditions. Thus advertisers may want to bid extra to get to right-hand side positions 1 and 2, hoping to be promoted to a top spot.

I examined the bids for a random sample of 2425 auctions involving at least 5 ads each on a particular day. Solving the quadratic programming problems yields a set of minimal perturbations for each auction required to make that auction satisfy the SNE inequalities. For each auction I define the mean absolution deviation to be $\sum_{s=1}^{S} |d_s - 1|/S$, where S is the number of advertisers in the auction.

Figure 4 depicts a histogram of mean absolute deviations necessary to satisfy the SNE inequalities; as it can be seen the deviations tend to be quite small, with the average absolute deviation of the perturbations being 5.8

Psn	raw LB	raw UB	pert LB	pert UB	price
1	1.19	∞	1.19	∞	0.48
2	1.29	2.26	1.15	2.26	0.60
3	0.60	1.66	1.19	1.48	0.48
4	0.72	0.30	0.60	0.61	0.23
5	1.68	1.79	1.47	1.49	0.40
6	0.23	0.84	0.34	0.74	0.07
7	1.32	0.83	1.08	1.19	0.24
8	0.05	1.63	0.05	1.33	0.21

Table 2: Bounds on values

percent and the median being 4.8 percent. Very few of the mean absolute deviations are larger than 10 percent. I conclude that relatively small perturbations are required to make the observations consistent with the SNE models. Since the NE inequalities are weaker than the SNE inequalities, the required perturbation for consistency with Nash equilibrium would be even smaller.

We can use the procedure for estimating the bounds on v described in section 5 to determine empirically the relationship between the bids and the values. For example, Table 2 shows the "raw" upper and lower bounds on values for a particular keyword calculated by using the observed incremental cost along with the upper and lower bounds on value calculated using the perturbed values from the quadratic program. The last column is the price of the click. In this example, the lower bounds sometimes exceed the upper bounds for the raw data, but the perturbed data satisfy the bounds by construction. The prices are not necessarily monotone due to the quality adjustment but the price times quality adjustment (not shown) is always monotone.

As can be seen from the table, the estimated value of a click to these bidders appears to be somewhere around a dollar and the advertisers are paying around fifty cents a click.

Appendix: Bayes-Nash equilibrium

Since the ad auction game that motivated this study runs continuously, it is reasonable to assume that the bidders are able to extract enough information about other bidders' behavior to find their way to a Nash equilibrium. However, it is also of interest to examine the Bayes-Nash equilibria, which might be appropriate in a game with substantially less information.

As it turns out the analysis is a straightforward variation of the classical analysis of a simple auction. To see this, let us first review the classical analysis. Let v be the value of a particular bidder, $P(v)$ the probability that he wins the auction, and $p(v)$ his expected payment. The bidder's objective is to maximize expected surplus $S(v) = vP(v) - p(v)$.

In the position auction context, we let $P_{(1)}(v)$ be the probability that the player has the highest bid, $P_{(2)}(v)$ the probability that the player with value v has the second-highest bid and so on. If there are 3 positions, the surplus becomes

$$S(v) = v[P_1(v)x_1 + P_2(v)x_2 + P_3(v)x_3] - p(v).$$

The first term is the expected surplus to a bidder with value v, recognizing that it gets x_1 clicks if it ends up in the first position, x_2 clicks if it ends up in the second position, and so on, with each click being worth v. In a simple auction, the value of coming in second is zero. In a position auction, the value of coming in second is vx_2.

Define $H(v) = P_{(1)}(v)x_1 + P_{(2)}(v)x_2 + P_{(3)}(v)x_3$ and write the surplus as

$$S(v) = vH(v) - p(v).$$

It is not hard to see that $H(v)$ has the relevant properties of a CDF. It is monotone, since it is a weighted sum of monotone functions. Furthermore if v_L and v_U are the upper and lower bounds on v, $H(v_L) = 0$ and $H(v_U) = x_1 =$ a constant.

All of the standard properties of a simple auction carry over to the position auction, including revenue neutrality, the derivation of the optimal reserve price, and so on. Hence the Bayes-Nash equilibrium of a position auction is a straightforward generalization of the Bayes-Nash equilibrium of a simple auction.

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